Continuity & differentiability of a function of single variables

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(Defn.) Continuity:- The function $f(x)$ is said to be continuous when $x=a$ if $f(x)$ possesses a definite limit as x tends to the value a from either side and each of these limits is equal to $f(a)$;

$$\text{i.e. } \underset{x\to a-0}{Lt.} f(x) = f(a) = \underset{x\to a+0}{Lt.} f(x).$$

(Defn.) Continuity in an interval:- A function $f(x)$ is said to be continuous in the closed interval $[a,b]$, i.e., $a \leq x \leq b$, if it is continuous at every point of the open interval $]a,b[$, i.e. $a < x < b$, and if $f(a+0)$ exists and is equal to $f(a)$, and $f(b-0)$ exists and is equal to $f(b)$.

Defn. (Differentiability):- A function $f(x)$ defined at a point a and in a certain neighbourhood of a is said to be differentiable at a if the limits

$$\underset{h\to 0}{Lt.} \frac{f(a+h)-f(a)}{h} \text{ and } \underset{h\to 0}{Lt.} \frac{f(a-h)-f(a)}{-h}$$

both exist and have the same value.

The value is called the differential coefficient of $f(x)$ at a and is denoted by $f'(a)$.

(Theorem) If a function is differentiable at a point, then prove that it must be continuous at that point.

Proof:- Let $f(x)$ be differentiable at $x=a$.

Then, by defn.

$$\underset{h\to 0}{Lt.} \frac{f(a+h)-f(a)}{h} = \underset{h\to 0}{Lt.} \frac{f(a-h)-f(a)}{-h} = A$$

$\therefore |f(a+h)-f(a)| = |h|[A+\epsilon]$, where $\epsilon \to 0$ as $h \to 0$

i.e. $f(a+h)-f(a) \to 0$ as $h \to 0$

i.e. $f(a+h) \to f(a)$ as $h \to 0$ — (1)

Also, $|f(a-h)-f(a)| = |h|[A+\epsilon_1]$, where $\epsilon_1 \to 0$ as $h \to 0$

i.e. $f(a-h)-f(a) \to 0$ as $h \to 0$

i.e. $f(a-h) \to f(a)$ as $h \to 0$ — (2)

From (1) and (2), we have

$$\underset{h\to 0}{Lt}\, f(a+h) = f(a) = \underset{h\to 0}{Lt}\, f(a-h).$$

Hence the function is Continuous at $x=a$.

(Theorem):- Show that Continuity is a weaker Condition on a function than differentiability.

or, Prove that the Converse of this theorem is not necessarily true.

Proof:- Let us examine the Continuity and the differentiability of the function $f(x)=|x|$ at $x=0$

The function $f(x)$ is Said to be Continuous at $x=0$ if

$$\underset{h\to 0}{Lt}\, f(0+h) = f(0) = \underset{h\to 0}{Lt}\, f(0-h).$$

Now, $\underset{h\to 0}{Lt}\, f(0+h) = \underset{h\to 0}{Lt}\, |0+h| = \underset{h\to 0}{Lt}\, |h| = \underset{h\to 0}{Lt}\, h = 0.$

$f(0) = |0| = 0.$

$\underset{h\to 0}{Lt}\, f(0-h) = \underset{h\to 0}{Lt}\, |0-h| = \underset{h\to 0}{Lt}\, |-h| = \underset{h\to 0}{Lt}\, h = 0.$

$$\therefore \underset{h\to 0}{Lt}\, f(0+h) = f(0) = \underset{h\to 0}{Lt}\, f(0-h).$$

Hence, the function $f(x)=|x|$ is Continuous at $x=0$.

$$\frac{\quad}{x} = \frac{|x|}{x} = \pm 1,$$

According as x is $\pm$ve.

$\therefore$ Hence $\lim_{x \to 0} \frac{f(x)-f(0)}{x}$ does not exist.

So, $f(x)$ is not differentiable at $x=0$.

Thus the function under Considera-tion is Continuous at $x=0$ but not differentiable at $x=0$.

(Theorem):- If a function is Continuous in a closed interval, Prove that it is bounded in that interval.

Proof:- Let the function $f(x)$ be Continuous in the closed interval $[a, b]$.

Let the interval $[a, b]$ be divided into a finite no. of sub-intervals, say 'n' such that the oscillation of $f(x)$ $< \epsilon$ in each sub-interval.

Let the dividing Point be $x_0 = a, x_1, x_2 \dots \dots x_{n-1}, x_n = b$.

Considering the first sub-interval $[a, x_1]$, we have

$$|f(x)| = |f(a) + f(x) - f(a)|$$

$$\text{or, } |f(x)| \leq |f(a)| + |f(x) - f(a)|$$

$$\text{or } |f(x)| < |f(a)| + \epsilon$$

In Particular when $x = x_1$,

$$|f(x_1)| < |f(a)| + \epsilon \quad \text{—— (1)}$$

Again, Considering the Second sub-interval $[x_1, x_2]$, we have

$$|f(x)| = |f(x_1) + f(x) - f(x_1)|$$

$$\text{or, } |f(x)| \leq |f(x_1)| + |f(x) - f(x_1)|$$

or $|f(x)| < |f(x_1)| + \epsilon$

or $|f(x)| < |f(a)| + \epsilon + \epsilon$, [by equn. of 1]

or $|f(x)| < |f(a)| + 2\epsilon$

In Particular, when $x = x_2$,

$$|f(x_2)| < |f(a)| + 2\epsilon$$

By proceeding in this way, when x lies in the nth sub-interval $[x_{n-1}, x_n]$, we have

$$|f(x)| < |f(a)| + n\epsilon \quad \text{——— (2)}$$

Since the inequality (2) is clearly satisfied in the whole interval $[a, b]$, therefore $f(x)$ is bounded in $[a, b]$.

Hence the theorem is Proved

Theorem:- If $f(x)$ be Continuous in the closed interval $[a, b]$, Prove that it attains its bounds at least once in this interval.

Proof:- We know that a function which is continuous in an interval is bounded.

Therefore $f(x)$ must be bounded in $[a, b]$.

Let the least upper bound and the glb of $f(x)$ be M and m respectively,

Now it would be Proved that there are at least two Point α & β in $[a, b]$ such that $f(\alpha) = M$, & $f(\beta) = m$.

First suppose that $f(x) \neq M$, for any value of x in $[a, b]$.

$\therefore M - f(x) \neq 0$, for any value of x in $[a, b]$

Let us now Consider the function

$$\phi(x) = \frac{1}{M - f(x)}$$

Since $f(x)$ is continuous in $[a,b]$, therefore $M-f(x)$ is also continuous in $[a,b]$.

$\therefore \dfrac{1}{M-f(x)}$, that is, $\phi(x)$.

is also continuous in $[a,b]$, as $M-f(x) \neq 0$.

So $\phi(x)$ is bounded in $[a,b]$.

Let G be the lub of $\phi(x)$ in $[a,b]$.

By the property of bound, we have

$$\phi(x) \leqslant G,$$

$$\text{or, } \frac{1}{M-f(x)} \leqslant G$$

$$\text{or } M-f(x) \geqslant \frac{1}{G}$$

$$\text{or, } f(x) < M - \frac{1}{G}, \text{ for all } x \text{ in } [a,b].$$

This contradicts the fact that M is the lub of $f(x)$ in $[a,b]$.

Hence our supposition that M is not attained by the function $f(x)$ at any point x in $[a,b]$ is wrong.

Thus $f(x)$ must attain M at least once in $[a,b]$.

Similarly it can be shown that the greatest lower bound m is attained by $f(x)$ at least once in $[a,b]$.

Hence the theorem is completely proved.